Homework 2, due 2/11

Only your **four** best solutions will count towards your grade.

1. Let $U \subset \mathbf{C}^n$ be open and connected, and let $f : U \to \mathbf{C}$ be holomorphic, not identically zero. Show that the complement $U \setminus Z(f)$ of the zero set is connected and dense in U.

2. Consider $f : \mathbf{C}^2 \to \mathbf{C}$ given by

$$f(z_1, z_2) = z_1^3 z_2 + z_1 z_2 + z_1^2 z_2^2 + z_1^2 + z_1 z_2^3.$$

   Find an explicit decomposition $f = g \cdot h$ given by the Weierstrass Preparation Theorem.

3. Show that in a decomposition $f = f_1 f_2 \cdots f_k$ of $f \in \mathcal{O}_{\mathbf{C}^n,0}$ into irreducibles, the f_i are unique up to changing the order, and multiplying by units to complete the proof that $\mathcal{O}_{\mathbf{C}^n,0}$ is a UFD.

4. An analytic set $X \subset \mathbf{C}^n$ is irreducible if we cannot write $X = X_1 \cup X_2$ as a union of analytic sets with $X_i \neq X$. One defines irreducible analytic germs analogously. Give an example of an irreducible analytic set that does not define irreducible analytic germs at every point, and an analytic set that is not irreducible, but whose germs are all irreducible.

5. Let $f : B_{1,1}(0) \to B_{1,1}(0)$ be a biholomorphism of the bidisk $B_{1,1}(0) \subset \mathbf{C}^2$, such that $f(0) = 0$. Use the Schwarz lemma to show that up to permuting the coordinates, and a rotation, we can assume that the Jacobian $J(f)(0)$ is the identity.

6. Let $f : B_{1,1}(0) \to B_{1,1}(0)$ be a biholomorphism such that $J(f)(0)$ is the identity. By considering the iterates f^k, show that f is the identity map. Deduce that $B_{1,1}(0)$ and the ball $D = \{z : \|z\| < 1\}$ are not biholomorphic.